

Generalized Secret Sharing using Permutation Ordered Binary System

Binu. V. P¹ and A. Sreekumar¹

Cochin University of Science and Technology, Cochin 22, India

Abstract. Secret sharing is a method of dividing a secret among n participants and allows only qualified subset to reconstruct the secret and hence provides better reliability and availability of secret data. In the generalized secret sharing scheme, a monotone access structure of the set of participants is considered. The access structure specifies a qualified subset of participants who can reconstruct the secret from their shares. Generalized secret sharing schemes can be efficiently implemented by using (n, n) scheme. We have developed an efficient (n, n) scheme using Permutation Ordered Binary (POB) number system which is then combined with cumulative arrays to obtain a generalized secret sharing scheme.

1 Introduction

Secret sharing schemes are important tool used in security protocols. Originally motivated by the problem of secure key storage by Shamir[26], secret sharing schemes have found numerous other applications in cryptography and distributed computing. Threshold cryptography[11], access control[24], secure multi party computation[3][9][10], attribute based encryption[13][5], generalized oblivious transfer[31][27], visual cryptography [23] etc ... are the significant areas of development using the secret sharing techniques.

In secret sharing, the secret is divided among n participants in such a way that only designated subset of participants can recover the secret, but any subset of participants which is not a designated set cannot recover the secret. A set of participants who can recover the secret is called an *access structure*, or *authorized set*, and a set of participants which is not an authorized set is called an *unauthorized set* or *forbidden set*. Let $\mathcal{P} = P_i | i = 1, 2, \dots, n$ be the set of participants and the secret be K . The set of all secret is represented by \mathcal{K} . The set of all shares S_1, S_2, \dots, S_n is represented by \mathcal{S} . The participants set is partitioned into two classes.

1. The class of authorized sets \mathcal{A} is called the *access structure*.
2. The class of unauthorized sets $\mathcal{A}^c = 2^{\mathcal{P}} \setminus \mathcal{A}$

We assume that $\mathcal{P}, \mathcal{K}, \mathcal{S}$ are all finite sets and there is a probability distribution on \mathcal{K} and \mathcal{S} . We use $H(\mathcal{K})$ and $H(\mathcal{S})$ to denote the entropy of \mathcal{K} and \mathcal{S} respectively.

In a secret sharing scheme there is a special participant called *Dealer* $\mathcal{D} \notin \mathcal{P}$, who is trusted by everyone. The dealer chooses a secret $K \in \mathcal{K}$ and the shares S_1, S_2, \dots, S_n corresponding to the secret is generated. The shares are then distributed privately to the participants through a secure channel.

In the secret reconstruction phase, participants of an access set pool their shares together and recover the secret. Alternatively participants could give their shares to a combiner to perform the computation for them. If an unauthorized set of participants pool their shares they cannot recover the secret. Thus a secret sharing scheme for the access structure \mathcal{A} is the collection of two algorithms:

Distribution Algorithm: This algorithm has to be run in a secure environment by a trustworthy party called Dealer. The algorithm uses the function f , which for a given secret $K \in \mathcal{K}$ and a participant $P_i \in \mathcal{P}$, assigns a set of shares from the set \mathcal{S} that is $f(K, P_i) = S_i \subseteq \mathcal{S}$ for $i = 1, \dots, n$.

$$f : \mathcal{K} \times \mathcal{P} \implies 2^{\mathcal{S}}$$

Recovery Algorithm: This algorithm has to be executed collectively by cooperating participants or by the combiner, which can be considered as a process embedded in a tamper proof module and all participants have access to it. The combiner outputs the generated result via secure channels to cooperating participants. The combiner applies the function

$$g : \mathcal{S}^t \implies \mathcal{K}$$

, to calculate the secret. For any authorized set of participants $g(S_1, \dots, S_t) = K$ if $P_1, \dots, P_t \subseteq \mathcal{A}$. If the group of participant belongs to an unauthorized set, the combiner fails to compute the secret.

A secret sharing scheme is called perfect if for all sets B , $B \subset \mathcal{P}$ and $B \notin \mathcal{A}$, if participants in B pool their shares together they cannot reduce their uncertainty about S . That is, $H(K) = H(K|S_B)$, where S_B denote the collection of shares of the participants in B . It is known that for a perfect secret sharing scheme $H(S_i) \geq H(K)$. If $H(S_i) = H(K)$ then the secret sharing scheme is called ideal.

An access structure \mathcal{A}_1 is *minimal* if $\mathcal{A}_2 \subset \mathcal{A}_1$ and $\mathcal{A}_2 \in \mathcal{A}$ implies that $\mathcal{A}_2 = \mathcal{A}_1$. Only *monotone access structure* is considered for the construction of the scheme in which $\mathcal{A}_1 \in \mathcal{A}$ and $\mathcal{A}_1 \subset \mathcal{A}_2$ implies $\mathcal{A}_2 \in \mathcal{A}$. The collection of minimal access sets uniquely determines the access structure. The access structure is the closure of the minimal access set. The access structure \mathcal{A} in terms of minimal access structure is represented by \mathcal{A}_{min} .

For an access structure \mathcal{A} , the family of unauthorized sets $\mathcal{A}^c = 2^{\mathcal{P}} \setminus \mathcal{A}$ has the property that given an unauthorized set $B \in \mathcal{A}^c$ then any subset $C \subset B$ is also an unauthorized set. An immediate consequence of this property is that for any access structure \mathcal{A} , the set of unauthorized sets can be uniquely determined by its *maximal set*. We use \mathcal{A}_{max}^c to denote the representation of \mathcal{A}^c in terms of maximal set.

For all $B \in \mathcal{A}$. If $|B| \geq t$ then the access structure corresponds to a (t, n) threshold scheme. In the (t, n) threshold scheme t or more participant can reconstruct the secret.

Section 2 gives survey of secret sharing schemes. Secret sharing schemes realizing general access structures are mentioned in section 3. Cumulative array implementation of generalized secret sharing is mentioned in section 4. The POB system is introduced in section 5. The proposed scheme and conclusions are given in section 6 and section 7.

2 Secret Sharing Schemes

The idea of *secret sharing* is to start with a secret, and divide it into pieces called *shares* or *shadows* which are distributed amongst users such that the pooled shares of authorized subsets of users allow reconstruction of the original secret.

Development of secret sharing scheme started as a solution to the problem of safeguarding cryptographic keys by distributing the key among n participants and t or more of the participants can recover it by pooling their shares. Thus the authorized set is any subset of participants containing more than t members. This scheme is denoted as (t, n) *threshold scheme*.

The following are the two fundamental requirements of any secret sharing scheme.

- **Recoverability:** Authorized subset of participants should be able to recover the secret by pooling their shares.
- **Privacy:** Unauthorized subset of participants should not learn any information about the secret.

The notion of a threshold secret sharing scheme is independently proposed by Shamir [26] and Blakley [6] in 1979. Since then much work has been put into the investigation of such schemes. Linear constructions were most efficient and widely used. A threshold secret sharing scheme is called perfect, if less than t shares give no information about the secret. Shamir's scheme is perfect while Blakley's scheme is non perfect. Both the Blakley and the Shamir constructions realize t -out-of- n shared secret schemes. However, their constructions are fundamentally different.

Shamir's scheme is based on polynomial interpolation over a finite field. It uses the fact that we can find a polynomial of degree $t - 1$ given t data points. To generate a polynomial $f(x) = \sum_{i=0}^{t-1} a_i x^i$, a_0 is set to the secret value and the coefficients a_1 to a_{t-1} are assigned random values in the field. The value $f(i)$ is given to the user i . When t out of n users come together they can reconstruct the polynomial using Lagrange interpolation and hence obtain the secret.

Blakley's secret sharing scheme has a different approach and is based on hyperplane geometry. To implement a (t, n) threshold scheme, each of the n users is given a hyper-plane equation in a t dimensional space over a finite field such that each hyperplane passes through a certain point. The intersection point of

these hyperplanes is the secret. When t users come together, they can solve the system of equations to find the secret.

McEliece and Sarwate [21] made an observation that Shamir's scheme is closely related to Reed-Solomon codes [25]. The error correcting capability of this code can be translated into desirable secret sharing properties.

Let $(\alpha_1, \alpha_2, \dots, \alpha_{r-1})$ be a fixed list of the non zero elements in a finite field F with r elements. In one form of Reed-Solomon coding, an information word $a = (a_0, a_1, \dots, a_{k-1})$, $a_i \in F$ is encoded into code word $D = (D_1, D_2, \dots, D_{r-1})$, where $D_i = \sum_{j=0}^{k-1} a_j \alpha_i^j$. The secret is $a_0 = -\sum_{i=1}^{r-1} D_i$, while the pieces of the secret are D_i 's. If given h shares but t of these are in error. Then by applying errors and erasures decoding algorithm it is possible to recover D and a , provided that $h - 2t \geq k$. This shows that if t pieces have been tampered, the secret can still be accessed by legitimate users provided that at least $k + t$ valid pieces are available. In the case of a (k, n) threshold scheme, the opponent must tamper $\lfloor (n - k)/2 \rfloor$ pieces to ensure that the secret is inaccessible.

Karnin et al [17] realize threshold schemes using linear codes. Massey [20] introduced the concept of minimal code words, and provided that the access structure of a secret sharing scheme based on a $[n, k]$ linear code is determined by the minimal codewords of the dual code.

Number theoretic concepts are also introduced for threshold secret sharing scheme. The Mingotte scheme [22] is based on modulo arithmetic and **Chinese Remainder Theorem (CRT)**. A special sequence of integers called Mingotte sequence is used here. Let n be an integer $n \geq 2$, and $2 \leq k \leq n$. A (k, n) Mingotte sequence is a sequence of pairwise coprime positive integers $p_1 < p_2 < \dots < p_n$ such that $\prod_{i=0}^{k-2} p_{n-i} < \prod_{i=1}^k p_i$. The shares are generated using the sequence. The secret is reconstructed by solving the set of congruence equation using CRT.

The Mingotte's scheme is not perfect. A perfect scheme based on CRT is proposed by Asmuth and Bloom [1]. They also uses a special sequence of pairwise coprime positive integers $p_0, p_1 < \dots < p_n$ such that $p_0 \cdot \prod_{i=0}^{k-2} p_{n-i} < \prod_{i=1}^k p_i$.

Kothari [18] gave a generalized threshold scheme. A secret is represented by a scalar and a linear variety is chosen to conceal the secret. A linear function known to all trustees is chosen and is fixed in the beginning, which is used to reveal the secret from the linear variety. The n shadows are hyperplanes containing the linear variety. Moreover the hyperplanes are chosen to satisfy the condition that, the intersection of less than t of them results in a linear variety which projects uniformly over the scalar field by the linear functional used for revealing the secret. The number t is called the threshold. Thus as more shadows are known more information is revealed about the linear variety used to keep the secret, however, no information is revealed until the threshold number of shadows are known. He had shown that Blakley's scheme and Karin's scheme are equivalent and provided algorithms to convert one scheme to another. He also stated that the schemes are all specialization of generalized linear threshold scheme.

Brickell [7] also give a generalized notion of Shamir and Blackleys schemes. The basic secret sharing scheme mentioned is as follows.

The secret is an element in some finite field $\mathbb{GF}(q)$. The dealer chooses a vector $a = (a_0, \dots, a_t)$ for some t , where each $a_j \in \mathbb{GF}(q)$, and a_0 is the secret. Denote the participants by P_i , for $1 \leq i \leq n$. For each P_i , the dealer will pick a t -dimensional vector v_i over $\mathbb{GF}(q)$. All of the vectors v_i , for $1 \leq i \leq n$ will be made public. The share that the dealer gives to P_i will be $S_i = v_i \cdot a$. Let e_i denote the i 'th t -dimensional unit coordinate vector (i.e. $e_1 = (1, 0, \dots, 0)$). The participants in \mathcal{P} can determine the secret if the subspace $\langle v_{i_1}, \dots, v_{i_k} \rangle$ contains e_1 . The participants in \mathcal{P} receive no information about the secret if the subspace $\langle v_{i_1}, \dots, v_{i_k} \rangle$ does not contain e_1 .

Researchers have investigated (t, n) threshold secret sharing extensively. Threshold schemes that can handle more complex access structures have been described by Simmons [28] like weighted threshold schemes, hierarchical scheme, compartmental secret sharing etc. They were found a wide range useful of applications.

3 Generalized Secret Sharing Schemes

In the previous section, we mentioned that any t of the n participants should be able to determine the secret. A more general situation is to specify exactly which subsets of participants should be able to determine the secret and which subset should not. In this section we give the secret sharing constructions based on generalized access structure.

Shamir [26] discussed the case of sharing a secret between the executives of a company such that the secret can be recovered by any three executives, or by any executive and any vice-president, or by the president alone. This is an example of *hierarchical secret sharing* scheme. The Shamir's solution for this case is based on an ordinary $(3, m)$ threshold secret sharing scheme. Thus, the president receives three shares, each vice-president receives two shares and, finally, every executive receives a single share.

The above idea leads to the so-called weighted (or multiple shares based) threshold secret sharing schemes. In these schemes, the shares are pairwise disjoint sets of shares provided by an ordinary threshold secret sharing scheme. Benaloh and Leichter have proven in [4] that there are access structures that can not be realized using such scheme. The theorem and proof with an example stated by them is given below.

Theorem 1. *There exist monotone access structure for which there is no threshold scheme.*

Proof. Consider the access structure \mathcal{A} defined by the formula $\mathcal{A}_0 = \{AB, CD\}$, and assume that a threshold scheme is to be used to divide a secret value S among A, B, C , and D such that only those subsets of A, B, C, D which are in \mathcal{A} can reconstruct S .

Let a, b, c , and d respectively denote the weight (number of shares) held by each of A, B, C , and D . Since A together with B can compute the secret, it must be the case that $a + b \geq t$ where t is the value of the threshold. Similarly, since C and D can together compute the secret, it is also true that $c + d \geq t$. Now

assume without loss of generality that $a \geq b$ and $c \geq d$. (If this is not the case, the variables can be renamed.) Since $a + b \geq t$ and $a \geq b, a + a \geq a + b \geq t$. So $a \geq t/2$. Similarly, $c \geq t/2$. Therefore, $a + c \geq t$. Thus, A together with C can reconstruct the secret value S . This violates the assumption of the access structure.

Several researchers address this problem and introduced secret sharing schemes realizing the general access structure. The most efficient and easy to implement scheme was Ito, Saito, Nishizeki's [14] construction. It is based on Shamir's scheme. The idea is to distribute shares to each authorized set of participants using multiple assignment scheme where more than one share is assigned to a participant if he belongs to more than one minimal authorized subset.

A simple scheme mentioned by Beimel [2] in which the secret $S \in \{0, 1\}$ and let \mathcal{A} be any monotone access structure. The dealer shares the secret independently for each authorized set $B \in \mathcal{A}$, where $B = \{P_{i1}, \dots, P_{il}\}$. The Dealer chooses $l - 1$ random bits r_1, \dots, r_{l-1} . Compute $r_l = S \oplus r_1 \oplus r_2 \oplus \dots \oplus r_{l-1}$, and the Dealer distributes share r_j to P_{ij} . For each set $B \in \mathcal{A}$, the random bits are chosen independently and each set in \mathcal{A} can reconstruct the secret by computing the exclusive-or of the bits given to the set. The unauthorized set cannot do so.

The disadvantage with multiple share assignment scheme is that the share size depends on the number of authorized set that contain P_j . A simple optimization is to share the secret S only for minimal authorized sets. Still this scheme is inefficient for access structures in which the number of minimal set is big (Eg: $(n/2, n)$ scheme). The share size grows exponentially in this case.

Benaloh and Leichter [4] developed a secret sharing scheme for an access structure based on monotone formula. This generalizes the multiple assignment scheme of Ito, Saito and Nishizeki [14]. The idea is to translate the monotone access structure into a monotone formula. Each variable in the formula is associated with a trustee in \mathcal{P} and the value of the formula is *true* if and only if the set of variables which are true corresponds to a subset of \mathcal{P} which is in the access structure. This formula is then used as a template to describe how a secret is to be divided into shares.

The monotone function contains only AND and OR operator. To divide secret S into shares such that P_1 or P_2 can reconstruct S . In this case P_1 and P_2 can simply both be given values S . If P_1 and P_2 need to reconstruct secret then P_1 can be given value S_1 and P_2 can be given value S_2 such that $S = S_1 + S_2 \bmod m$, ($0 \leq S \leq m$), s_1 is chosen randomly from \mathbb{Z}_m , S_2 is $(S - S_1) \bmod m$.

More exactly, for a monotone authorized access structure \mathcal{A} of size n , they defined the set $\mathcal{F}_{\mathcal{A}}$ as the set of formula on a set of variables $\{v_1, v_2, \dots, v_n\}$ such that for every $\mathcal{F} \in \mathcal{F}_{\mathcal{A}}$, the interpretation of \mathcal{F} with respect to an assignment of the variables is true if and only if the true variables correspond to a set $A \in \mathcal{A}$. They have remarked that such formula can be used as templates for describing how a secret can be shared with respect to the given access structure. Because the formula can be expressed using only \wedge operators and \vee operators, it is sufficient to indicate how to "split" the secret across these operators.

Thus, we can inductively define the shares of a secret S with respect to a formula \mathcal{F} as follows:

$$Shares(S, F) = \begin{cases} (S, i), & \text{if } F = v_i, 1 \leq i \leq n; \\ \bigcup_{i=1}^k Shares(S, F_i), & \text{if } F = F_1 \vee \dots \vee F_k; \\ \bigcup_{i=1}^k Shares(s_i, F_i), & \text{if } F = F_1 \wedge \dots \wedge F_k, \end{cases}$$

where, for the case $F = F_1 \wedge F_2 \wedge \dots \wedge F_k$, we can use any (k, k) -threshold secret sharing scheme for deriving some shares s_1, \dots, s_k corresponding to the secret S and, finally, the shares as $I_i = \{s | (s, i) \in Shares(S, F)\}$, for all $1 \leq i \leq n$, where, F is an arbitrary formula in the set \mathcal{F}_A .

Brickell [8] developed some ideal schemes for generalized access structure using vector spaces. Stinson [30] introduced a monotone circuit construction based on monotone formula and also the construction based on public distribution rules. Benaloh's scheme was generalized by Karchmer and Wigderson [16] who showed that if an access structure can be described by a small monotone span program then it has an efficient scheme. The proposed generalized secret sharing scheme make use of the cumulative arrays for the generalized secret sharing which is given in the next section.

4 Cumulative Secret Sharing Scheme

Cumulative schemes were first introduced by Ito et al [14] and then used by several authors to construct a general scheme for arbitrary access structures. Simmons [28] proposed cumulative map, Jackson [15] proposed a notion of cumulative array. Ghodosi et al [12] introduced simpler and more efficient scheme and also introduced capabilities to detect cheaters. Generalized cumulative arrays in secret sharing is introduced by Long [19].

Definition 1 Let \mathcal{A} be a monotone authorized access structure on a set of participants \mathcal{P} . A cumulative scheme for the access structure \mathcal{A} is map $\alpha : \mathcal{P} \rightarrow 2^S$, where S is some set. such that for any $\mathcal{A} \subseteq P$,

$$\bigcup_{P_i \in \mathcal{A}} \alpha(P_i) = S$$

The scheme can be written as a $|\mathcal{P}| \times |S|$ array $M = [m_{ij}]$, where row i of the matrix M is indexed by $p_i \in P$ and column j of the matrix M is indexed by an element $s_j \in S$, such that $m_{ij} = 1$ if and only if P_i is given s_j , otherwise $m_{ij} = 0$.

Definition 2 Let \mathcal{A} be an access structure over the set of participants $\mathcal{P} = \{P_1, \dots, P_n\}$ and $\mathcal{A}_{min} = \{\mathcal{A}_1, \dots, \mathcal{A}_l\}$ is the set of all minimal set of \mathcal{A} . Then the **incident array** of \mathcal{A} is a $l \times n$ Boolean matrix $I_A = [a_{ij}]$ defined by,

$$a_{ij} = \begin{cases} 1 & \text{if } P_j \in \mathcal{A}_i \\ 0 & \text{if } P_j \notin \mathcal{A}_i \end{cases}$$

for $1 \leq j \leq n$ and $1 \leq i \leq l$

Definition 3 Let $\mathcal{A}_{max}^c = \{B_1, \dots, B_m\}$ be the set of all maximal unauthorized sets. The **cumulative array** $C_{\mathcal{A}}$ for \mathcal{A} is an $n \times m$ matrix $C_{\mathcal{A}} = [b_{ij}]$, where each row of the matrix is indexed by a participant $P_i \in \mathcal{P}$ and each column is indexed by a maximal unauthorized set $B_j \in \mathcal{A}_{max}^c$, such that the entries b_{ij} satisfy the following:

$$b_{ij} = \begin{cases} 0 & \text{if } P_i \in B_j \\ 1 & \text{if } P_i \notin B_j \end{cases}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$

It is noted that following theorem is true and proved in [12].

Theorem 2. If α_i is the i 'th row of the cumulative array $C_{\mathcal{A}}$, then $\alpha_{i1} + \dots + \alpha_{it} = \vec{1}$ if and only if $\{P_{i1}, \dots, P_{it}\} \in \mathcal{A}$

cumulative scheme of [14] uses Shamir's threshold [26] scheme where as Blakley's scheme is used by [15]. A simple scheme using cumulative array and Karnin-Greene-Hellman threshold scheme [17] proposed by Ghodosi et al [12] is given below.

The Scheme

Let $\mathcal{A}_{min} = \mathcal{A}_1 + \dots + \mathcal{A}_{\ell}$ be a monotone access structure over the set of participants $\mathcal{P} = P_1, \dots, P_n$. Let $\mathcal{A}_{max}^c = B_1 + \dots + B_m$ be the set of maximal unauthorized subsets. The share distribution and reconstruction phases are given below.

Share Distribution Phase

1. The dealer D , constructs the $n \times m$ cumulative array $C_{\mathcal{A}} = [b_{ij}]$, where n is the number of participants and m is the cardinality of \mathcal{A}_{max}^c
2. \mathcal{D} used Karnin-Greene-Hellman(m,m) threshold scheme [17] to generate m shares $S_j, 1 \leq j \leq m$.
3. \mathcal{D} gives shares S_j privately to participant P_i if and only if $b_{ij} = 1$.

Secret Reconstruction Phase

1. The secret can be recovered by every access set using the modular addition over \mathbb{Z}_q

Example 1. Let $n = 4$ and $\mathcal{A}_{min} = \{\{1, 2\}, \{3, 4\}\}$. In this case, we obtain that $\mathcal{A}_{max}^c = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$ and $m = 4$.

The cumulative array for the access structure \mathcal{A} is,

$$C_{\mathcal{A}} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

In this case, $S_1 = \{s_3, s_4\}$, $S_2 = \{s_1, s_2\}$, $S_3 = \{s_2, s_4\}$ and $S_4 = \{s_1, s_3\}$, where s_1, s_2, s_3, s_4 are the shares of a (4, 4)-threshold secret sharing scheme.

5 Permutation Ordered Binary System(POB)

The POB system is developed by Sreekumar et al [29] for the efficient storage and computations associated with share generation and reconstruction. Simple ex-or operations are used for the reconstruction of secret. The share generation algorithm is also linear and depends on the size of the secret. The shares generated are 1 bit less than the secret but still provides the same level of security and hence a reduction in storage space can be achieved. The POB system can be used to implement an (n, n) scheme very efficiently.

5.1 POB construction

The POB number system is represented by $POB(n, r)$, where n and r are positive integers and $n \geq r$. In this number system, we represent all integers in the range $0, \dots, \binom{n}{r} - 1$, as a binary string, say $B = b_{n-1}b_{n-2} \dots b_0$, of length n , and having exactly r 1s.

Each digit of this number, say, b_j is associated with its position value, given by

$$b_j \cdot \binom{j}{p_j}, \text{ where, } p_j = \sum_{i=0}^j b_i,$$

and the value represented by the POB-number B , denoted by $V(B)$, will be the sum of position values of all of its digits.

i.e.,

$$V(B) = \sum_{j=0}^{n-1} b_j \cdot \binom{j}{p_j} \quad (1)$$

It can be proved that, since exactly $\binom{n}{r}$ such binary strings exist, each number will have a distinct representation. In order to emphasize that a binary string, $B = b_{n-1}b_{n-2} \dots b_0$ is a POB-number, we denote the same by using the suffix ' p '. For example, 001110100_p is a POB(9, 4) number represented by 33. However, such a string, regarded as a binary number will have a decimal value of 116.

It is proved that the POB-representation is unique in the sense that the binary representation of a POB-number is unique. The value of a POB-number, $V(B)$ of $B = b_{n-1}b_{n-2} \dots b_0$ computed by the formula (1) given above, produces distinct values in the range $0, \dots, \binom{n}{r} - 1$. Efficient algorithms are also developed to convert POB values into POB number and vice versa.

5.2 (n,n) scheme using POB

It is noted that efficient (n, n) schemes are the building blocks of secret sharing schemes having more generalized monotone access structure. Karnin [17] et al

developed an unanimous consent scheme which is used in the Benaloh's and Leichter scheme [4]. Ito et al [14] used Shamir's (n, n) threshold scheme. POB system can be used for developing an efficient (n, n) scheme which is secure and reliable. The details of construction is given as Algorithm.

Algorithm 1 (Sharing a secret among n blocks)

Input: A single byte string $K = K_1K_2K_3 \dots K_8$.

Output : n shares S_1, S_2, \dots, S_n of length 7 bits.

Step 1. Let A_1, A_2, \dots, A_n be null strings of length 9 bits.

Step 2. Randomly assign $n-2$ POB(9,4)-numbers one for each of $A_i, 2 \leq i \leq n-1$.

Let $r = \left\lceil \frac{V(A_2)+1}{14} \right\rceil$

Step 3. The input string K is expanded to T by inserting one bit at position r .

Compute the binary string $T = T_1T_2 \dots T_9$

$$T_i = \begin{cases} K_i, & \text{if } i < r \\ K_{i-1}, & \text{if } i > r \\ 0, & \text{if } i = r \text{ and } K \text{ is even parity} \\ 1, & \text{if } i = r \text{ and } K \text{ is odd parity} \end{cases}$$

Step 4. Let $W = T \oplus A_2 \oplus A_3 \oplus \dots \oplus A_{n-1}$

Step 5. Let $W = W_1W_2 \dots W_9$

$noOfOne = 0;$

For $i = 1$ to 9 do

if $(W_i = 1)$ then

$noOfOne = noOfOne + 1;$

if $(noOfOne \text{ is odd}) A_{1i} = 1;$

else $A_{1i} = 0;$

Step 6. Randomly assign the rest null bits of A_1 to 0 or 1, let A_1 consists of four 1s and five 0s.

Step 7. Compute $A_n = W \oplus A_1$

Step 8. For $i = 1$ to n do

$S_i = V(A_i).$

Algorithm 2 (Recover the secret information)

Input : n shares S_1, S_2, \dots, S_n of length 7 bits each.

Output: The secret information $K = K_1K_2K_3 \dots K_8$.

Step 1. Let A_1, A_2, \dots, A_n be the POB-numbers corresponding to S_1, S_2, \dots, S_n respectively and $r = \left\lceil \frac{S_2+1}{14} \right\rceil$

Compute $T = A_1 \oplus A_2 \oplus A_3 \oplus \dots \oplus A_n$

Let $T = T_1T_2 \dots T_9$

Step 2. For $i = 1$ to 8 do
 if $(i \geq r)$ $j = i + 1$;
 else $j = i$;
 $K_i = T_j$.

Step 3. The recovered secret is $K = K_1K_2K_3 \dots K_8$

Example 2. For a (5, 5) threshold scheme, secret $K = 10110110$ is taken.

Randomly assign five 0s and four 1s to 3 rows $\{A_2, A_3, A_4\}$. Therefore,

$$\begin{aligned} A_2 &= 101100010, \\ A_3 &= 010101001, \text{ and} \\ A_4 &= 110010100. \end{aligned}$$

Let the random number $r = \left\lceil \frac{V(A_2)+1}{14} \right\rceil = \left\lceil \frac{102}{14} \right\rceil = 8$.

The expanded string T as per step 3, of Algorithm 1 is $T = 101101110$

Step 4. Computes $W = 100110001$,

by Step 5., $A_1 = 1**01***0$, and

by step 6., A_1 becomes $= 110010100$ by Step 7, $A_5 = 010100101$

The shares are the indices: 113, 101, 48, 113, 46. All the 5 shares are listed below:

$$\begin{aligned} S_1 &= 1110001, \\ S_2 &= 1100101, \\ S_3 &= 0110000, \\ S_4 &= 1110001, \text{ and} \\ S_5 &= 0101110. \end{aligned}$$

Recovery: Compute $T = A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5$, and get 101101110. Deleting the 8th bit, we get secret as $K = 10110110$.

6 Proposed Generalized Secret Sharing Scheme

The proposed scheme make use of (n,n) scheme using POB and cumulative arrays to efficiently share a secret according to a generalized access structure.

Algorithm 3 Generalized Secret Sharing using POB

Input: Access structure corresponds to a secret sharing scheme.

Output: Shares for each participants corresponds to the given access structure.

Step 1. Find the maximal unauthorized set \mathcal{A}_{max}^c corresponds to the given access structure.

Step 2. The dealer D , constructs the $n \times m$ cumulative array $C_A = [b_{ij}]$, where n is the number of participants and m is the cardinality of \mathcal{A}_{max}^c .

Step 3. D uses (m, m) POB scheme to generate m shares $S_j, 1 \leq j \leq m$.

Step 4. D gives shares S_j privately to participant P_i if and only if $b_{ij} = 1$.

Algorithm 4 *Secret Reconstruction using POB*

Input: Shares corresponds to the participants.

Output: Shared secret corresponds to the authorized set or error.

Step 1. From the shares generate the POB number.

Step 2. The secret can be reconstructed by ex-oring the shares corresponds to an authorized set.

Step 3. For an unauthorized set the algorithm gives an error else the secret is returned.

7 Conclusion

In this paper we explored the secret sharing schemes realizing the general access structure. Several schemes are proposed but the share size is a major concern. Number of shares received by the participant grows exponentially in generalized secret sharing. We have proposed a scheme with cumulative arrays and (n, n) threshold scheme using POB. The size of the share is smaller in this case and also the secret can be easily reconstructed by simple XOR operation. An 8 bit secret can be shared with a share of 7 bit size. The probability of guessing the share reduces as the size of the secret to be shared increases.

References

1. Asmuth, C., Bloom, J.: "A modular approach to key safeguarding". Information Theory, IEEE Transactions on 29(2), 208–210 (1983)
2. Beimel, A.: "Secret-sharing schemes: a survey". In: Coding and Cryptology, pp. 11–46. Springer (2011)
3. Ben-Or, M., Goldwasser, S., Wigderson, A.: "Completeness theorems for non-cryptographic fault-tolerant distributed computation". In: Proceedings of the twentieth annual ACM symposium on Theory of computing. pp. 1–10. ACM (1988)
4. Benaloh, J., Leichter, J.: "Generalized secret sharing and monotone functions". In: Advances in Cryptology CRYPTO88. pp. 27–35. Springer (1990)
5. Bethencourt, J., Sahai, A., Waters, B.: "Ciphertext-policy attribute-based encryption". In: Security and Privacy, 2007. SP'07. IEEE Symposium on. pp. 321–334. IEEE (2007)
6. Blakley, G.R., et al.: "Safeguarding cryptographic keys". In: Proceedings of the national computer conference. vol. 48, pp. 313–317 (1979)
7. Brickell, E.F.: "Some ideal secret sharing schemes". Journal of Combinatorial Mathematics and Combinatorial Computing 9(2), 105–113 (1989)
8. Brickell, E.F., Davenport, D.M.: "On the classification of ideal secret sharing schemes". Journal of Cryptology 4(2), 123–134 (1991)
9. Chaum, D., Crépeau, C., Damgård, I.: "Multiparty unconditionally secure protocols". In: Proceedings of the twentieth annual ACM symposium on Theory of computing. pp. 11–19. ACM (1988)
10. Cramer, R., Damgård, I., Maurer, U.: "General secure multi-party computation from any linear secret-sharing scheme". In: Advances in Cryptology EUROCRYPT 2000. pp. 316–334. Springer (2000)

11. Desmedt, Y., Frankel, Y.: "Shared generation of authenticators and signatures". In: *Advances in Cryptology CRYPTO91*. pp. 457–469. Springer (1992)
12. Ghodosi, H., Pieprzyk, J., Safavi-Naini, R., Wang, H.: "On construction of cumulative secret sharing schemes". In: *Information Security and Privacy*. pp. 379–390. Springer (1998)
13. Goyal, V., Pandey, O., Sahai, A., Waters, B.: "Attribute-based encryption for fine-grained access control of encrypted data". In: *Proceedings of the 13th ACM conference on Computer and communications security*. pp. 89–98. ACM (2006)
14. Ito, M., Saito, A., Nishizeki, T.: "Secret sharing scheme realizing general access structure". *Electronics and Communications in Japan (Part III: Fundamental Electronic Science)* 72(9), 56–64 (1989)
15. Jackson, W.A., Martin, K.M.: "Cumulative arrays and geometric secret sharing schemes". In: *Advances in Cryptology AUSCRYPT'92*. pp. 48–55. Springer (1993)
16. Karchmer, M., Wigderson, A.: "On span programs". In: *Structure in Complexity Theory Conference, 1993., Proceedings of the Eighth Annual*. pp. 102–111. IEEE (1993)
17. Karnin, E., Greene, J., Hellman, M.: "On secret sharing systems". *Information Theory, IEEE Transactions on* 29(1), 35–41 (1983)
18. Kothari, S.: "Generalized linear threshold scheme". In: *Advances in Cryptology*. pp. 231–241. Springer (1985)
19. Long, S., Pieprzyk, J., Wang, H., Wong, D.S.: "Generalised cumulative arrays in secret sharing". *Designs, Codes and Cryptography* 40(2), 191–209 (2006)
20. Massey, J.L.: "Minimal codewords and secret sharing". In: *Proceedings of the 6th Joint Swedish-Russian International Workshop on Information Theory*. pp. 276–279. Citeseer (1993)
21. McEliece, R.J., Sarwate, D.V.: "On sharing secrets and Reed-Solomon codes". *Communications of the ACM* 24(9), 583–584 (1981)
22. Mignotte, M.: "How to share a secret". In: *Cryptography*, pp. 371–375. Springer (1983)
23. Naor, M., Shamir, A.: "Visual cryptography". In: *Advances in Cryptology EURO-CRYPT'94*. pp. 1–12. Springer (1995)
24. Naor, M., Wool, A.: "Access control and signatures via quorum secret sharing". *Parallel and Distributed Systems, IEEE Transactions on* 9(9), 909–922 (1998)
25. Reed, I.S., Solomon, G.: "Polynomial codes over certain finite fields". *Journal of the Society for Industrial & Applied Mathematics* 8(2), 300–304 (1960)
26. Shamir, A.: "How to share a secret". *Communications of the ACM* 22(11), 612–613 (1979)
27. Shankar, B., Srinathan, K., Rangan, C.P.: "Alternative protocols for generalized oblivious transfer". In: *Distributed Computing and Networking*, pp. 304–309. Springer (2008)
28. Simmons, G.J.: "An introduction to shared secret and/or shared control schemes and their application". *Contemporary Cryptology: The Science of Information Integrity* pp. 441–497 (1992)
29. Sreekumar, A., Sundar, S.B.: "An Efficient Secret Sharing Scheme for n out of n scheme using POB-number system". *Hack. in 2009* p. 33 (2009)
30. Stinson, D.R.: "An explication of secret sharing schemes". *Designs, Codes and Cryptography* 2(4), 357–390 (1992)
31. Tassa, T.: "Generalized oblivious transfer by secret sharing". *Designs, Codes and Cryptography* 58(1), 11–21 (2011)